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the b 's being real, and r being the rank of (c_{gh}) .

$$\therefore c_{gh} = \sum_{i=1}^r b_{ig} b_{ih}$$

$$\therefore W_{gh} = \sum_{m=0}^{\infty} a_m \left(\sum_{i=1}^r b_{ig} b_{ih} \right) = \sum_{p_1, \dots, p_r} C_{p_1, \dots, p_r} \left(b_{1g} b_{1h} \right)^{p_1} \left(b_{2g} b_{2h} \right)^{p_2} \left(b_{rg} b_{rh} \right)^{p_r},$$

C standing for
 p_1, \dots, p_r

$$\frac{|\Sigma p|}{\Pi |p|} a_{\Sigma p}.$$

$\therefore (W_{gh})$ is the matrix of the quadratic form

$$(1) \quad \sum_{p_1, \dots, p_r=0, 1, \dots} C_{p_1, \dots, p_r} \left(\sum_{j=1}^r b_{1j}^{p_1} \dots b_{rj}^{p_r} y_j \right)^2,$$

which is also a positive or vanishing form, since the series in fact is absolutely convergent for arbitrary values of y_1, \dots, y_r .

$\therefore |W_{gh}| \geq 0$, which is the first part of the generalized theorem.

Suppose now that $|W_{gh}| = 0$, so that the form (1) is only semi-definite. The C 's being now all positive, it follows that any form obtained from (1) by giving to the C 's other positive values will be semi-definite. But $w_{gh} = \exp(kc_{gh})$, which differs from W_{gh} only in the values of the (positive) coefficients, k being positive. $\therefore (w_{gh}^k)$ is the matrix of a semi-definite form. Therefore

$$(2) \quad |w_{gh}^k| = 0, \quad (k = 1, 2, \dots).$$

Now every minor of $|w_{gh}|$ of the type $\begin{vmatrix} w_{gg} & w_{gh} \\ w_{gh} & w_{hh} \end{vmatrix}$ must be ≥ 0 . If all such are > 0 , then for every g and h ($g \neq h$), w_{gh} has an absolute value $< \sqrt{w_{gg} w_{hh}}$. \therefore the leading term of the determinant $|w_{gh}|$ is greater in absolute value than any other term and > 1 . $\therefore \lim_{k \rightarrow \infty} |w_{gh}^k| = \infty$, which contradicts (2). \therefore for some g and h ($g \neq h$), $w_{gg} w_{hh} = w_{gh}^2$.

That is $c_{gg} + c_{hh} = 2c_{gh}$, or $\int_0^1 (\phi_g - \phi_h)^2 dx = 0$. Hence, $\phi_g(x) = \phi_h(x)$ on (01).

422. Proposed by O. S. ADAMS, Coast and Geodetic Survey, Washington, D. C.

Prove that

$$\int_0^1 \int_0^1 f(xy) (1-x)^{m-1} y^m (1-y)^{n-1} dx dy = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \int_0^1 f(z) (1-z)^{m+n-1} dz,$$

$f(xy)$ being an arbitrary function of xy .

SOLUTION BY THE PROPOSER.

Let us arrange the integral as follows:

$$I = \int_0^1 dy \int_0^1 f(xy) (y - xy)^{m-1} (1-y)^{n-1} y dx,$$

the integration with respect to x to be made first, considering y as a constant.

Let $z = xy$, then $dz = ydx$. This gives us

$$I = \int_0^1 dy \int_0^y f(z) (y-z)^{m-1} (1-y)^{n-1} dz.$$

Changing the order of integrations, we have

$$I = \int_0^1 f(z) dz \int_z^1 (y-z)^{m-1} (1-y)^{n-1} dy.$$

For a justification of this step see Bôcher's *Integral Equations*, p. 4, Dirichlet's Formula.

Now let

$$y - z = (1 - z)w, \quad dy = (1 - z)dw,$$

and

$$1 - y = (1 - z)(1 - w).$$

By substituting these results, we obtain

$$I = \int_0^1 f(z)(1 - z)^{m+n-1} dz \int_0^1 w^{m-1}(1 - w)^{n-1} dw = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \int_0^1 f(z)(1 - z)^{m+n-1} dz.$$

This problem is taken from Whittaker & Watson's *Modern Analysis*, p. 250. Given in *Math. Trip.*, 1894.

Note.—A second solution was received but no name signed to it. EDITOR.

423. Proposed by J. B. REYNOLDS, Lehigh University.

Show that the envelope of all circles with their centers on the circle $x^2 + y^2 = a^2$ and tangent to the x -axis is the two-arched epicycloid.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

The general form of equation of circles fulfilling the conditions of the problem is

$$f(x, y, x_1, r) = (x - x_1)^2 + (y - r)^2 - r^2 = 0, \quad (1)$$

with the conditional equation

$$\varphi(x_1, r, a) = x_1^2 + r^2 - a^2 = 0. \quad (2)$$

We are to find the envelope of the system of circles represented by (1), x_1, r being the variable parameters. Using the undetermined multiplier λ and the auxiliary equations

$$\frac{df}{dx_1} = \lambda \frac{d\varphi}{dx_1}, \quad (3)$$

$$\frac{df}{dr} = \lambda \frac{d\varphi}{dr}, \quad (4)$$

we have

$$-(x - x_1) = \lambda x_1, \quad (5)$$

and

$$-(y - r) - r = \lambda r. \quad (6)$$

We must eliminate x_1, r, λ from (1), (2), (5), and (6). Substituting the values of $x - x_1$, and $y - r$ from (5) and (6) in (1),

$$\lambda^2 x_1^2 + r^2(1 + \lambda)^2 = r^2. \quad (7)$$

Also eliminating x_1 from (2) and (7),

$$\lambda(2r^2 + a^2\lambda) = 0. \quad (8)$$

The value of $\lambda = 0$ is irrelevant, but the second factor in (8) gives

$$r^2 = -\frac{1}{2}a^2\lambda. \quad (9)$$

Substituting (9) in (2) gives

$$x_1^2 = \frac{a^2}{2}(\lambda + 2) \quad (10)$$

and (5) gives

$$x_1 = \frac{x}{1 - \lambda}, \quad (11)$$

which with (10) gives the cubic in λ

$$a^2\lambda^3 - 3a^2\lambda + 2(a^2 - x^2) = 0. \quad (12)$$

Again, (6) gives

$$r = \frac{y}{\lambda}, \quad (13)$$

and this with (9) gives the second cubic in λ ,

$$a^2\lambda^3 + 2y^2 = 0. \quad (14)$$